Advanced Algorithms — Exercise Set 5

- Submit on Gradescope by 10:00pm on **March 12th** (note the unusual due date).
- Feel free to discuss with others, but write up your own work.
- Half points on this exercise set are awarded for completion / effort. Use it to learn!

Problem 1 (Comparing Formulations)**.** Let $S_1 := \left\{x \in \{0,1\}^4 : 90x_1 + 35x_2 + 26x_3 + 25x_4 \leq 138\right\}$.

1. Let $S_2 = \left\{x \in \{0,1\}^4 : 2x_1 + x_2 + x_3 + x_4 \leq 3\right\}$, and let

$$S_3 = \left\{ x \in \{0,1\}^4 : \begin{array}{rcl} 2x_1 + x_2 + x_3 + x_4 & \leq & 3, \\ x_1 + x_2 + x_3 & \leq & 2, \\ x_1 + x_2 + x_4 & \leq & 2, \\ x_1 + x_3 + x_4 & \leq & 2. \end{array} \right\}.$$

   Show that $S_1 = S_2 = S_3$.

2. Let S_i' be the linear relaxation of the set S_i, in other words, where the $x \in \{0,1\}^4$ constraint is replaced by $x \in [0,1]^4$. Order the three sets S_i' by inclusion.

Problem 2 (Independent Set). Given a graph $G = (V, E)$, an *independent set* is a subset of vertices $S \subseteq V$ such that no two vertices in S share an edge. The goal is to find the largest such set. An integer program for the independent set problem is as follows:

$$\begin{aligned} \max \quad & \sum_{v \in V} x_v \\ \text{s.t.} \quad & x_u + x_v \leq 1 \qquad \forall \{u, v\} \in E, \\ & x_v \in \{0, 1\} \qquad \forall v \in V. \end{aligned}$$

1. Show that the integrality gap of the LP relaxation of this formulation on a graph with n vertices can be $2/n$. [Hint: consider a complete graph on n vertices]

2. Can you think of valid inequalities you can add to this Integer Program to strengthen the formulation?

Problem 3 (Max-3-SAT). In the Max-3-SAT problem, we are given a Boolean formula in conjunctive normal form (look this up if you don't remember what this means) where each clause contains exactly three literals (variables or their negations). There are n variables and m clauses. The goal is to find a truth assignment maximizing the number of satisfied clauses.

1. Formulate the Max-3-SAT problem as an integer linear program. Justify your formulation. [Hint: introduce a decision variable x_i for each of the n booleans, and a decision variable z_j for each of the m clauses.]

2. Relax the integrality constraints on the variables in your IP to obtain its linear relaxation. Show that the fractional optimal value is always equal to m.

Problem 4 (Duality)**.** Consider the following linear program (P):

$$
\begin{aligned}
\text{(P)} \qquad \max \quad & 3x_1 + 2x_2 - x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 4, \\
& 2x_1 - x_2 + 3x_3 \leq 5, \\
& -x_1 + x_2 + x_3 \leq 2, \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.
\end{aligned}
$$

1. Take the dual of (P) to obtain the dual program (D).
2. What is the relationship between the optimal objective value of (P) and (D)?
3. What is the relationship between the objective values of an integer solution to (P) and an integer solution to (D)?